Random matrix theory: a review and new results

Kei Tokita

Cybermedia Center, Graduate School of Science

and Graduate School of Frontier Biosciences

Osaka University, Toyonaka, Osaka 560-0043, Japan

Matrices with elements drawn randomly from statistical distributions are called "random matrices". It has been established that physical properties of many disordered systems (such as amorphous materials, a magnetic alloy like spin glasses, etc.) are determined by mathematical properties of random matrices, in particular, by their eigenvectors and eigenvalues [1,2,3]. One of the most famous results is the so-called Wigner's semicircular law [1] which states that the average density $\rho(x)$ of the eigenvalue x of a $N \times N$ symmetric real random matrix (a_{ij}) in the limit of matrix size $N \to \infty$ is

$$\rho(x) = \begin{cases} \dfrac{\sqrt{4\sigma^2 - x^2}}{2\pi\sigma^2} & \text{if } |x| < 2\sigma \\ 0 & \text{otherwise,} \end{cases}$$

where each element a_{ij} is drawn from independent identical distribution with zero odd-order moments, finite even-order moments and variance σ^2. This theorem was applied by May [5] to a linear stability analysis for a system with random interactions which exhibits a sharp transition from stable to unstable behavior when N (diversity) or the typical interaction strength σ (complexity) exceeds a critical value, the phenomenon of which is first discovered numerically by Gardner and Ashby [4]. I will first sketch out their results and the succeeding controversy on the stability of a large and complex ecosystem which is well known as the "paradox of ecology", and secondly give some recent examples of applications of random matrices to mathematical biology.

URL : `http://www.cp.cmc.osaka-u.ac.jp/~tokita/`

Email : `tokita@cmc.osaka-u.ac.jp`

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