Geometry, Mechanics, and their Applications
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Symmetry and conservation law of electrically charged particle in magnetic field

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Celebration
for the 70th birthday of
Iwai Toshihiro sensei

祝
古
希

Today I talk about “Extra Terms”.

おまけ

OMAKE = additional something

My results have already appeared in Prof. Kori's talk.

残
念

ZAN-NEN = regrettable

Introduction of myself

- My name is Shogo Tanimura.
- In past, I was an assistant (1995-1999) and an associate professor (2006-2011) working as a member of the group conducted by Iwai sensei at Kyoto University.
- My main concerns are foundation of quantum theory, dynamical system theory, and application of differential geometry to physics.

Plan of this talk

1. Raise and formulation of the problem
2. Answer in terms of Lagrangian formalism
3. Answer in terms of Hamiltonian formalism
4. Remaining problems

Raise of a problem 1/2

Magnetic field 2-form

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy, \qquad dB = 0$$

Vector potential 1-form

$$A = A_x dx + A_y dy + A_z dz, \qquad B = dA$$

Lagrangian and Hamiltonian involve the vector potential

$$L = \frac{1}{2m}\left({v_x}^2 + {v_y}^2 + {v_z}^2\right) + U(x, y, z)$$
$$+e\left(A_x v_x + A_y v_y + A_z v_z\right),$$

$$H = \frac{1}{2m}\left\{(p_x - eA_x)^2 + \left(p_y - eA_y\right)^2 + (p_z - eA_z)^2\right\}$$
$$+U(x, y, z)$$

Raise of a problem 2/2

Even if the magnetic field is invariant along a vector field u

$$\mathcal{L}_u B = 0 \text{ (Lie derivative),}$$

the vector potential 1-form itself is not invariant:

$$\mathcal{L}_u B = \mathcal{L}_u(dA) = 0, \qquad \text{but} \quad \mathcal{L}_u A \neq 0.$$

Lagrangian and Hamiltonian are not invariant under the action of u, either.

Can we find a conserved quantity associated to the vector field u ?

Free particle

Hamiltonian:

$$H = \frac{1}{2m}\{{p_x}^2 + {p_y}^2\}$$

Conserved quantities:

$$p_x, \qquad p_y, \qquad J := xp_y - yp_x$$

Charged particle

Hamiltonian:

$$H = \frac{1}{2m}\left\{(p_x - eA_x)^2 + \left(p_y - eA_y\right)^2\right\}$$

Magnetic field and vector potential:

$$B \coloneqq \partial_x A_y - \partial_y A_x$$

Kinetic momenta and their Poisson bracket:

$$\pi_x \coloneqq p_x - eA_x, \qquad \pi_y \coloneqq p_y - eA_y, \qquad \left\{\pi_x, \pi_y\right\}_{\mathrm{P}} = eB$$

Equations of motion:

$$\frac{dx}{dt} = \frac{1}{m}\pi_x, \qquad \frac{dy}{dt} = \frac{1}{m}\pi_y$$

$$\frac{d\pi_x}{dt} = \frac{e}{m}\pi_y B = eB\,\frac{dy}{dt}, \qquad \frac{d\pi_y}{dt} = -\frac{e}{m}\pi_x B = -eB\,\frac{dx}{dt}$$

Charged particle

Equations of motion:

$$\frac{d\pi_x}{dt} = eB\frac{dy}{dt}, \qquad \frac{d\pi_y}{dt} = -eB\frac{dx}{dt}$$

$$\frac{d}{dt}\left(x\pi_y - y\pi_x\right) = -xeB\frac{dx}{dt} - yeB\frac{dy}{dt} = -\frac{eB}{2}\frac{d}{dt}(x^2 + y^2)$$

Assume homogeneous magnetic field $B(x, y) = \text{constant}$

Conserved quantities:

$$\tilde{\pi}_x \coloneqq \pi_x - eBy,$$

$$\tilde{\pi}_y \coloneqq \pi_y + eBx,$$

$$\tilde{J} \coloneqq x\pi_y - y\pi_x + \frac{eB}{2}(x^2 + y^2)$$

Magnetic perturbation

Minimal gauge coupling:

$$p_x \longrightarrow \pi_x := p_x - eA_x$$
$$p_y \longrightarrow \pi_y := p_y - eA_y$$

$$H = \frac{1}{2m}\{p_x{}^2 + p_y{}^2\} \longrightarrow H = \frac{1}{2m}\left\{(p_x - eA_x)^2 + \left(p_y - eA_y\right)^2\right\}$$

But conserved quantities do not obey the naive minimal replacement rule:

$$F(\boldsymbol{r}, \boldsymbol{p}) \longrightarrow F(\boldsymbol{r}, \boldsymbol{p} - e\boldsymbol{A})$$

π_x is not conserved. Instead, $\tilde{\pi}_x := \pi_x \boxed{- eBy}$ is conserved.

$J = x\pi_y - y\pi_x$ is not conserved. Instead,

$$\tilde{J} = x\pi_y - y\pi_x \boxed{+ \frac{eB}{2}(x^2 + y^2)}$$

Extra terms

is conserved.

Statement of the problem

Minimal gauge coupling:

$$p_x \longrightarrow \pi_x \coloneqq p_x - eA_x$$

$$p_y \longrightarrow \pi_y \coloneqq p_y - eA_y$$

$$H = \frac{1}{2m}\{p_x{}^2 + p_y{}^2\} \longrightarrow H = \frac{1}{2m}\left\{(p_x - eA_x)^2 + \left(p_y - eA_y\right)^2\right\}$$

When a chargelss particle has a conserved quantity

$$F(\boldsymbol{r}, \boldsymbol{p}),$$

does a corresponding charged particle has a conserved quantity in the form

$$\tilde{F} = F(\boldsymbol{r}, \boldsymbol{p} - e\boldsymbol{A}) + W(\boldsymbol{r}, \boldsymbol{p})\ ?$$

Do we have a general rule for finding the extra term W ?

Monopole magnetic field

Spherically symmetric potential: $U(r)$

Vector potential: $A = A_x dx + A_y dy + A_z dz$

Magnetic field: $B = dA = g \sin\theta \, d\theta \wedge d\phi$

$$H = \frac{1}{2m}\left\{(p_x - eA_x)^2 + \left(p_y - eA_y\right)^2 + (p_z - eA_z)^2\right\} + U(r)$$

Kinetic momenta: $\boldsymbol{\pi} := \boldsymbol{p} - e\boldsymbol{A}$

Conserved quantity:

$$\tilde{\boldsymbol{J}} = \boldsymbol{r} \times \boldsymbol{\pi} - eg\frac{\boldsymbol{r}}{r}$$

Do we have a general rule for finding the extra term?

Lagrangian formalism

Noether theorem:

If under an infinitesimal transformation $\boldsymbol{q} \longmapsto \boldsymbol{q} + \delta\boldsymbol{q}$, the Lagrangian is quasi-invariant

$$\delta L = -\frac{dW(\boldsymbol{q}, t)}{dt},$$

then the system has a conserved quantity

$$F = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + W.$$

Magnetic field in Lagrangian

Minimal coupling

$$L = L_0 + eA_j\dot{r}_j$$

Assume that L_0 is invariant under an infinitesimal transformation $r_j \longmapsto r_j + \varepsilon u_j(\boldsymbol{r})$. Then

$$\delta\left(A_j\dot{r}_j\right) = \varepsilon\left\{\frac{\partial A_j}{\partial r_k}u_k\dot{r}_j + A_j\dot{u}_j\right\}$$

$$= \varepsilon\left\{\left(\frac{\partial A_j}{\partial r_k} - \frac{\partial A_k}{\partial r_j}\right)u_k\dot{r}_j + \frac{\partial A_k}{\partial r_j}u_k\dot{r}_j + A_k\dot{u}_k\right\}$$

$$= \varepsilon\left\{B_{kj}u_k\dot{r}_j + \frac{d}{dt}(A_k u_k)\right\}$$

On the other hand, $\delta L = \varepsilon\left\{\frac{\partial L}{\partial r_j}u_j + \frac{\partial L}{\partial \dot{r}_j}\dot{u}_j\right\} = \varepsilon\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_j}u_j\right)$

Geometric consideration

Let us to rewrite this term $B_{kj}u_k\dot{r}_j$

2-form $B = B_{kj}dr_k \otimes dr_j$

interior product $i_u B = B_{kj}u_k\, dr_j$ with $u = u_k \frac{\partial}{\partial r_k}$

homotopy formula: $(di_u + i_u d)B = \mathcal{L}_u B$: Lie derivative

Gauss' law for magnetic field: $dB = 0$

Assumption of symmetry: $\mathcal{L}_u B = 0$

Therefore, $di_u B = 0$.

Poincare's lemma tells $\exists W_u$ 0-form satisfying $i_u B = -dW_u$

Namely, we can write

$$B_{kj}u_k\dot{r}_j = -\frac{dW_u}{dt}$$

Geometric consideration

$$L = L_0 + eA_j\dot{r}_j$$

$$\delta r_j = \varepsilon u_j$$

$$\delta L_0 = 0$$

$$\delta L = \varepsilon \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_j} u_j\right) = \varepsilon \frac{d}{dt}\left(p_j u_j\right)$$

$$\delta\left(eA_j\dot{r}_j\right) = e\varepsilon\left\{B_{kj}u_k\dot{r}_j + \frac{d}{dt}\left(A_j u_j\right)\right\} = e\varepsilon \frac{d}{dt}\left(-W_u + A_j u_j\right)$$

By putting them together, we reach

$$\frac{d}{dt}\left(p_j u_j + eW_u - eA_j u_j\right) = \frac{d}{dt}\left\{\left(p_j - eA_j\right)u_j + eW_u\right\} = 0$$

Theorem (Main Result)

When a dynamical system L_0 is invariant under an action of vector field u, it has a Noether conservation quantity

$$F_u = \frac{\partial L_0}{\partial \dot{r}_j} u_j = p_j u_j.$$

If an applied magnetic field $B = dA$ is invariant, $\mathcal{L}_u B = 0$, there exists a function W_u such that $\boldsymbol{i_u B = -dW_u}$. Then the corresponding system in the magnetic field admits a conserved quantity

minimal coupling + extra term

$$\tilde{F}_u = (p_j - eA_j)u_j + eW_u.$$

This gives an answer to the problem proposed first.

Example: homogenous magnetic field

In $\mathbb{R}^2$, assume $B = Bdx \wedge dy$ with a constant B.

1) It is invariant under $u = \frac{\partial}{\partial x}$.

The equation $i_u B = Bdy = -dW_u$ has a solution $W_u = -By$.

The corresponding conserved quantity is

$$\tilde{F}_u = (p_j - eA_j)u_j + eW_u = \pi_x - eBy.$$

2) The magnetic field is invariant under $u = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

The equation $i_u B = -Bxdx - Bydy = -dW_u$ has a solution

$W_u = \frac{1}{2}B(x^2 + y^2)$.

The corresponding conserved quantity is

$$\tilde{F}_u = (p_j - eA_j)u_j + eW_u = x\pi_y - y\pi_x + \frac{1}{2}eB(x^2 + y^2).$$

Hamiltonian formalism (general)

Manifold M

Symplectic form ω (2-form, nondegenerated, $d\omega = 0$)

Hamiltonian $H \in C^\infty(M)$

Hamilton vector field X_f $\quad i_{X_f}\omega = -df \quad$ for $f \in C^\infty(M)$

Poisson bracket $\{f, g\} = X_g f = -X_f g$ for $f, g \in C^\infty(M)$

Symmetry: vector field X

$$\mathcal{L}_X \omega = 0, \qquad \mathcal{L}_X H = 0$$

Then, $0 = \mathcal{L}_X \omega = d i_X \omega + i_X d\omega = d(i_X \omega)$, therefore locally exists f such that $i_X \omega = -df$, namely $X = X_f$. Then

$$\{f, H\} = X_H f = -X_f H = -\mathcal{L}_{X_f} H = 0,$$

hence, f is a conserved quantity.

Magnetic field perturbation

Manifold $M = T^*\mathbb{R}^3$

Minimal coupling in the symplectic form

$$\omega_0 = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz$$

$$\omega = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz + eB$$

Symmetry: vector field X

$$\mathcal{L}_X \omega_0 = 0, \qquad \mathcal{L}_X B = 0, \qquad \mathcal{L}_X H = 0$$

Then, locally exists F, W such that $i_X \omega_0 = -dF$ and $i_X B = -dW$. Then by putting $\tilde{F} = F + eW$, we have $i_X \omega = i_X(\omega_0 + eB) = -d(F + eW) = -d\tilde{F}$, and hence

$$\{\tilde{F}, H\}_\omega = -X_{\tilde{F}} H = -\mathcal{L}_X H = 0.$$

hence, f is a conserved quantity.

This equation reproduces the extra term for the Noether charge.

Remaining problem

Defining equations

$$i_X\omega = -df, \qquad \{f, g\} = X_g f$$

scalar function $\longrightarrow$ vector field

$f \longmapsto X_f$

$[X_f, X_g] = -X_{\{f,g\}}$

vector field $\longrightarrow$ scalar function

$X \longmapsto f_X$ modulo additional constant

$\{f_X, f_Y\} = -f_{[X,Y]} + c(X, Y)$

Cohomology of the Lie algebra.

If it is trivial, the set of functions $\{f_X\}$ are called momentum maps.

$i_u B = -dW_u$, $\tilde{F}_u = \langle p - eA, u\rangle + eW_u$. Is the cohomology of the Noether charges associated to a magnetic field trivial?

$$\{\tilde{F}_u, \tilde{F}_v\} = -\tilde{F}_{[u,v]} + eW_{[u,v]} + ei_u i_v B$$

Summary and future work

- We found a scheme for transforming a conserved quantity of a chargeless particle to a conserved quantity of a charged particle in a magnetic field that admits the same symmetry.

minimal coupling + extra term

$$F_u = \frac{\partial L_0}{\partial \dot{r}_j} u_j = p_j u_j \;\; \rightarrow \;\; \tilde{F}_u = (p_j - eA_j) u_j + eW_u$$

$$i_u B = -dW_u$$

defining equation for the extra term

- Cohomological structure
- Relation with Marsden-Weinstein reduction
- Laplace-Runge-Lenz vector
- Quantization

Thank you for your attention