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Online participation

Quantization of Constraint System and Homological Commutation Relation of Electric and Magnetic Fluxes

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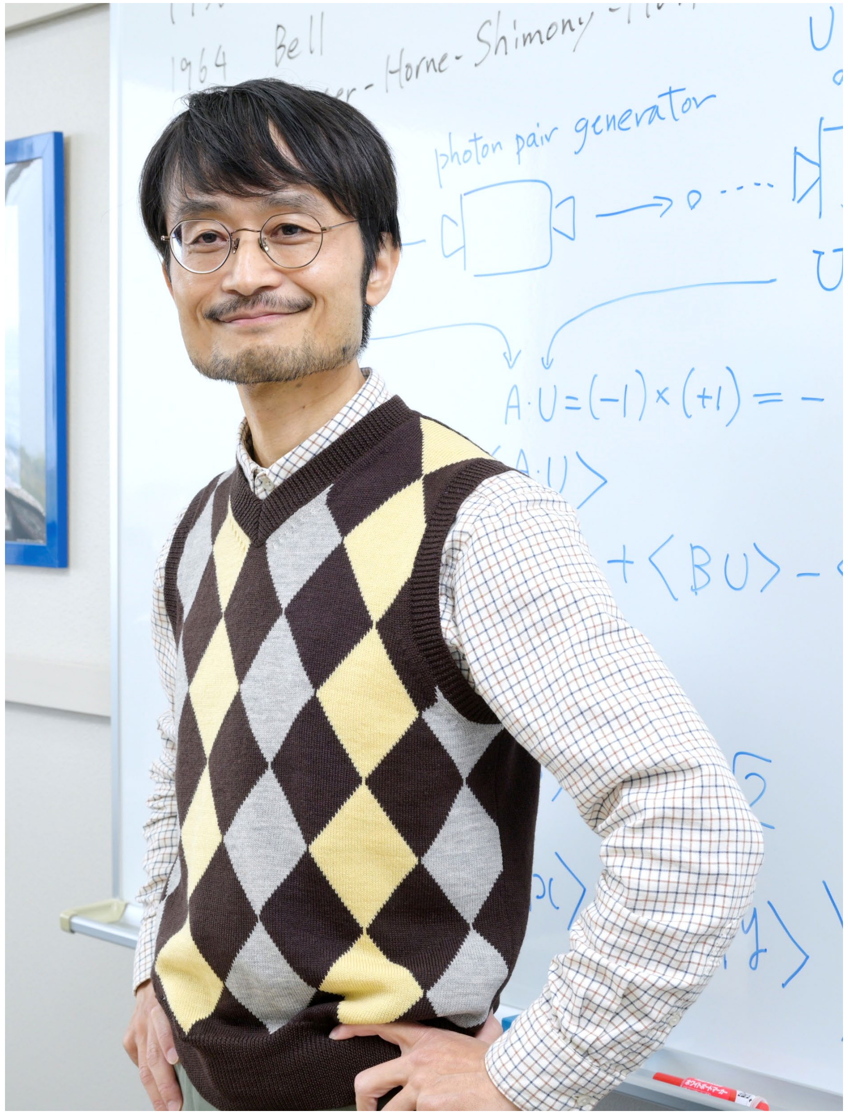
Introduction of myself

My name is Shogo Tanimura.

I am a theoretical physicist working in Japan, at Nagoya University.

My main research interests are quantum theory and application of differential geometry to physics.
Recently I am a member of a project to build AI for science.



My participations to GIQ

I attended the Third International Conference on Geometry, Integrability and Quantization (GIQ), which was held at Varna in 2001 and was organized by Dr. Ivailo M. Mladenov at that time, too. So, today is my second participation to the GIQ conference.

Main result presented today

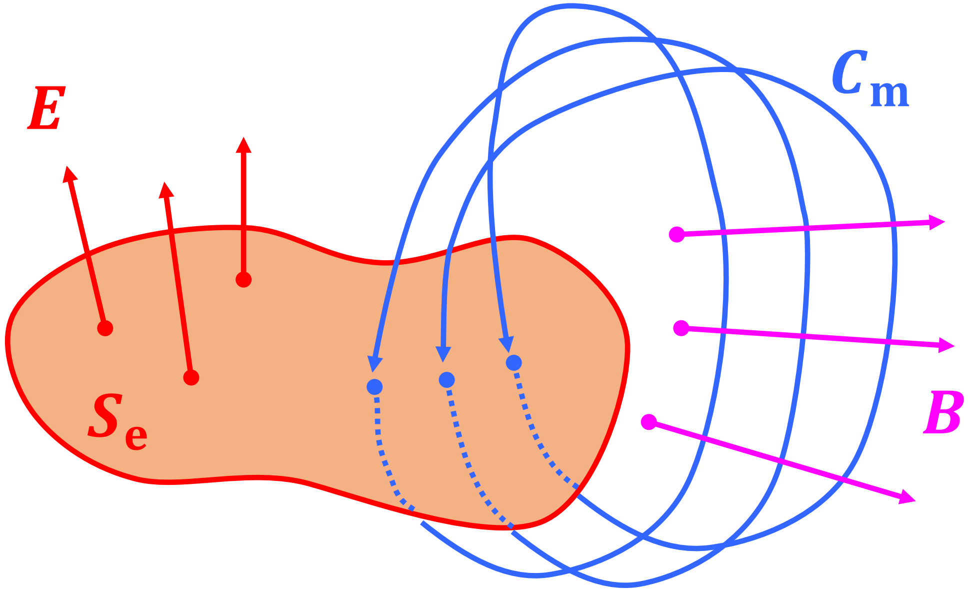
- Commutation relation: $[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}] = i\hbar\hat{\mathbf{1}} \cdot N(S_{\mathrm{e}}, S_{\mathrm{m}})$

- Electric flux operator: $\hat{\boldsymbol{\Phi}}_{\mathrm{e}} := \int_{S_{\mathrm{e}}} \hat{\boldsymbol{E}} \cdot \boldsymbol{n}\, d\sigma$

- Magnetic flux operator: $\hat{\boldsymbol{\Phi}}_{\mathrm{m}} := \int_{C_{\mathrm{m}}} \hat{\boldsymbol{A}} \cdot d\boldsymbol{r} = \int_{S_{\mathrm{m}}} \hat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\sigma$

- Crossing number: N

I prove this relation by applying Dirac's method of quantization of constraint system to the electromagnetic field.



Constraint system

Dynamical system (M, ω, H)

manifold M, symplectic 2-form ω, Hamiltonian H

Constraint: $J = (J_1, \cdots, J_k)$; $J_i : M \to \mathbb{R}$; $J_i(x) = c_i$

They are conserved quantities, $\{J_i, H\}_{\mathrm{Piosson}} = 0$

Constraint manifold $J^{-1}(c) \subseteq M$ for $c \in \mathbb{R}^k$

How to construct a Hamiltonian dynamical system on $J^{-1}(c)$?

(Rather trivial) example

Lagrangian in a configuration space $(x, y) \in \mathbb{R}^2$

$$L = \frac{1}{2} m\dot{x}^2 + a\dot{y}$$

Canonical momenta

$$p_x := \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y := \frac{\partial L}{\partial \dot{y}} = a = \text{constant}$$

Constraint $J := p_y = a$, or $\phi := p_y - a = 0$

$$H := p_x\dot{x} + p_y\dot{y} - L = \frac{1}{m}p_x^2 + a\dot{y} - \left(\frac{1}{2m}p_x^2 + a\dot{y}\right) = \frac{1}{2m}p_x^2$$

Reduced phase space $(x, p_x) \in \mathbb{R}^2$ ((y, p_y) are irrelevant.)

Three methods for constraint system

1. Dirac method (use of submanifold):
Change the notation $\phi_i := J_i - c_i$ $(i = 1, \cdots, k)$ and introduce other constraints ϕ_i $(i = k+1, \cdots, m)$ to define a set of constraints $\Phi = (\phi_1, \cdots, \phi_m)$ such that the matrix $T_{ij} \coloneqq \{\phi_i, \phi_j\}_{\mathrm{P}}$ $(i, j = 1, \cdots, m)$ becomes invertible, then, the 2-form ω restricted on the constrained submanifold $S \coloneqq \Phi^{-1}(0) \subseteq M$ is non-degenerated and hence it defines a reduced Hamiltonian system $(S, \omega|_S, H|_S)$, which is equivalent to the Dirac bracket

$$\{A, B\}_{\mathrm{D}} \coloneqq \{A, B\}_{\mathrm{P}} - \{A, \phi_i\}_{\mathrm{P}} \left(T^{-1}\right)_{ij} \{\phi_j, B\}_{\mathrm{P}}$$

Three methods for constraint system

2. Marsden-Weinstein method (use of quotient):
The constraint quantities (momentum maps) J_i $(i = 1, \cdots, k)$ take values in a dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group G and act on M by Hamiltonian vectors. For each value $c \in \mathfrak{g}^*$, the isotropy group G_c that preserves $c \in \mathfrak{g}^*$ by co-adjoint action, and then on the quotient manifold $Q := J^{-1}(c)/G_c$, a unique symplectic 2-form ω_c is induced and hence it defines a reduced Hamiltonian system (Q, ω_c, H).

Three methods for constraint system

3. BRST (Becchi, Rouet, Stora and Tyutin) symmetry (use of nilpotent algebra):
Introduce Grassmann odd variables c(ghost), $\bar{c}$(anti-ghost) and even variable b. Define the BRST transformation that is nilpotent and define the gauge-fixing term by BRST-exact form, which is additional term to the original Lagrangian. Claim that observable physical quantities are BRST-closed form.

Electromagnetic field as a dynamical system

- Space-time $\mathbb{R}^4$ with the Lorentzian metric $(+,-,-,-)$
- Gauge field (1-form): $A = A_\mu dx^\mu$ $(\mu = 0, 1, 2, 3)$
- Electromagnetic field (2-form): $F = dA = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$

  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

  electric field $E^i = F_{0i}$, magnetic field $B^i = -\frac{1}{2}\varepsilon^{ijk} F_{jk}$
- Lagrangian: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu = \frac{1}{2} F \wedge * F - A \wedge J$
- Gauge invariance: $A \longmapsto A + d\varphi$
- Euler-Lagrange equation (Maxwell equation): $\partial_\mu F^{\mu\nu} = J^\nu$

Constraints in electromagnetic dynamics 1/2

- Canonical momenta:

$$\Pi^\mu := \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A_\mu\right)} = -F^{0\mu} \qquad (\Pi^i = E^i)$$

- Primary constraint: $\Pi^0 = -F^{00} = 0$
- Formal Legendre transformation gives the Hamiltonian

$$H = \Pi^\mu \partial_0 A_\mu - \mathcal{L}$$

$$= \frac{1}{2}\left(E^2 + B^2\right) - A^i J^i + \Pi^0 \partial_0 A_0 - (\partial_i \Pi^i - J^0) A_0$$

- Secondary constraint: $\partial_i \Pi^i - J^0 = 0$

Constraints in electromagnetic dynamics 2/2

- Additional constraint (Coulomb gauge): $\boldsymbol{\partial_i A^i = 0}$
- $\boldsymbol{\partial_i \Pi^i - J^0 = 0}$ implies the Poisson equation

$$\boldsymbol{\partial_i\left(-\partial_i A^0 - \partial_0 A^i\right) = -\partial_i \partial_i A^0 = J^0}$$

- Helmholtz decompositions:

$$\boldsymbol{A = A_\parallel + A_\perp, \qquad \mathrm{rot}\, A_\parallel = 0, \qquad \mathrm{div}\, A_\perp = 0}$$

$$\boldsymbol{E = E_\parallel + E_\perp, \qquad \mathrm{rot}\, E_\parallel = 0, \qquad \mathrm{div}\, E_\perp = 0}$$

- Constrained variables:

$$\boldsymbol{A^0 = -\Delta^{-1} J^0, \qquad A_\parallel = 0, \qquad \Pi^0 = 0, \qquad \Pi_\parallel = E_\parallel = -\mathrm{grad}\, A^0}$$

- Independent canonical variables: $\boldsymbol{A_\perp, E_\perp}$

Canonical quantization

- **Transversal components satisfies the canonical commutation relation** $(\hat{A}_k = -\hat{A}^k, \hat{\Pi}^k = -\hat{\Pi}_k = \hat{E}^k)$

$$-\left[\hat{A}^j(x,t), \hat{E}^k(y,t)\right] = i\hbar\left(\delta_{jk} - \frac{\partial_j \partial_k}{\Delta}\right)\delta^3(x-y)$$

Derivation of $[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}] = i\hbar\hat{\mathbf{1}} \cdot N$

- CCR

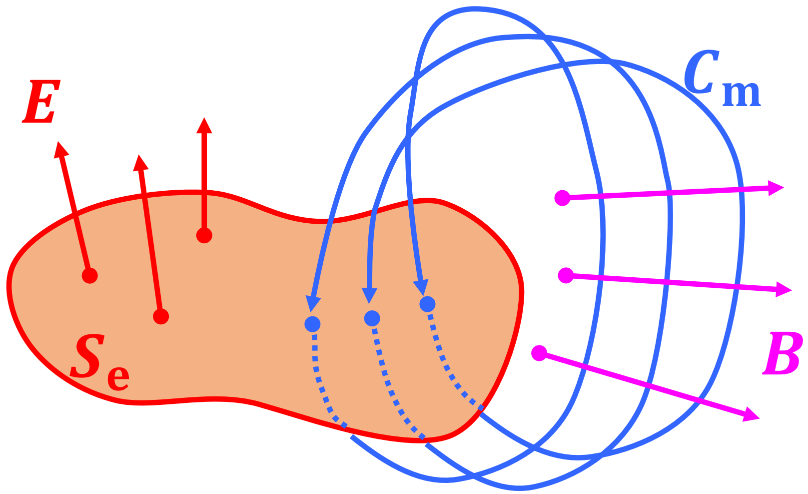
$$-\left[\hat{A}_{\perp}^{j}(\boldsymbol{x}, t), \hat{E}_{\perp}^{k}(\boldsymbol{y}, t)\right] = i\hbar\left(\delta_{jk} - \frac{\partial_j \partial_k}{\Delta}\right)\delta^3(\boldsymbol{x} - \boldsymbol{y})$$

- Line integral with respect to x and surface integral wrt y

$$\hat{\boldsymbol{\Phi}}_{\mathrm{m}} = \int_{C_{\mathrm{m}}} \hat{\boldsymbol{A}}_{\perp} \cdot d\boldsymbol{r} = \int_{S_{\mathrm{m}}} \hat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\sigma, \qquad \hat{\boldsymbol{\Phi}}_{\mathrm{e}} = \int_{S_{\mathrm{e}}} \hat{\boldsymbol{E}}_{\perp} \cdot \boldsymbol{n}\, d\sigma$$

yield the linking number:

$$-[\hat{\boldsymbol{\Phi}}_{\mathrm{m}}, \hat{\boldsymbol{\Phi}}_{\mathrm{e}}] = [\hat{\boldsymbol{\Phi}}_{\mathrm{e}}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}] = i\hbar\hat{\mathbf{1}} \cdot N(S_{\mathrm{e}}, S_{\mathrm{m}})$$



Homological invariance

- Homological deformation of surface does not change the magnetic flux since $\mathrm{div}\,\widehat{\boldsymbol{B}} = \mathbf{0}$:

$$\widehat{\boldsymbol{\Phi}}_{\mathrm{m}} = \int_{S_{\mathrm{m}}} \widehat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma} = \int_{S'_{\mathrm{m}}} \widehat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma}$$

- Inclusion of the longitudinal electric field changes the electric flux

$$\int_{S_{\mathrm{e}}} \widehat{\boldsymbol{E}}_{\perp} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma} = \int_{S'_{\mathrm{e}}} \widehat{\boldsymbol{E}}_{\perp} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma}, \qquad \int_{S_{\mathrm{e}}} \left(\widehat{\boldsymbol{E}}_{\perp} + \widehat{\boldsymbol{E}}_{\parallel}\right) \cdot \boldsymbol{n}\, d\boldsymbol{\sigma} \neq \int_{S_{\mathrm{e}}} \widehat{\boldsymbol{E}}_{\perp} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma}$$

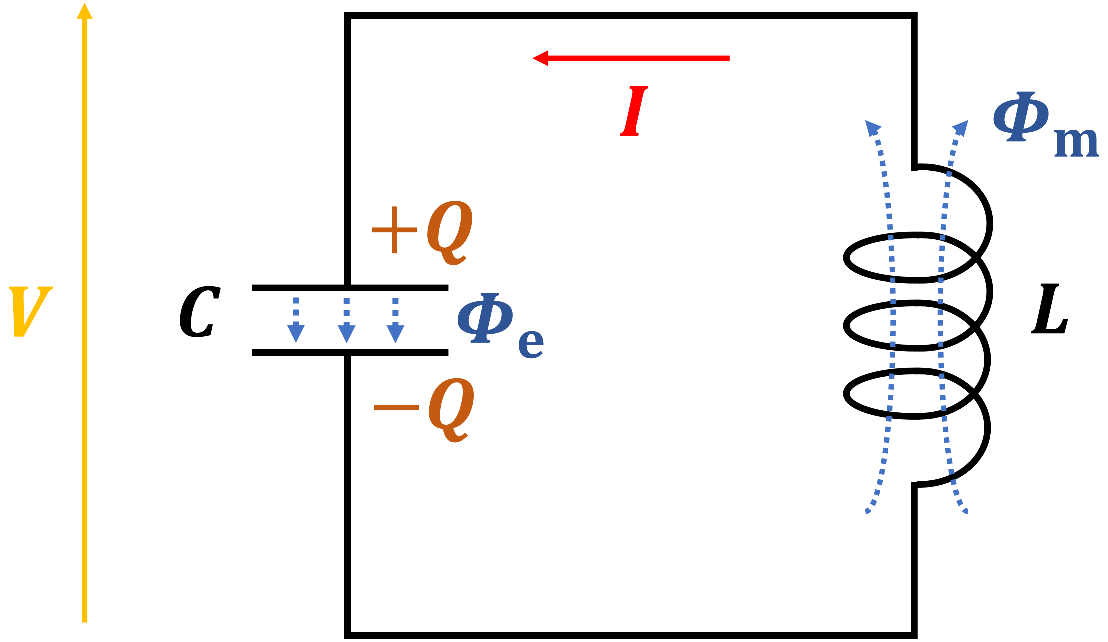
but does not change the flux-flux commutator since

$$\left[\widehat{A}^{j}_{\perp}(\boldsymbol{x}, \boldsymbol{t}), \widehat{E}^{k}_{\parallel}(\boldsymbol{y}, \boldsymbol{t})\right] = \mathbf{0}, \qquad \left(\boldsymbol{E}_{\parallel} = -\mathbf{grad}\, A^{0} = \mathbf{grad}\, \Delta^{-1} J^{0}\right)$$

- The flux-flux commutator $\left[\widehat{\boldsymbol{\Phi}}_{\mathrm{e}}, \widehat{\boldsymbol{\Phi}}_{\mathrm{m}}\right] = i\hbar\widehat{\mathbf{1}} \cdot N(S_{\mathrm{e}}, S_{\mathrm{m}})$ is homologically invariant as it is to be.

Application to LC circuit

- L=inductance, C=capacitance
- Inductance has a magnetic flux Φ_m associated with electric current I.
- Capacitance has an electric flux Φ_e associated with electric charge Q.



Classical LC circuit

- Constituent equations:

$$\Phi_{\mathrm{e}} = Q = CV, \qquad \Phi_{\mathrm{m}} = LI$$

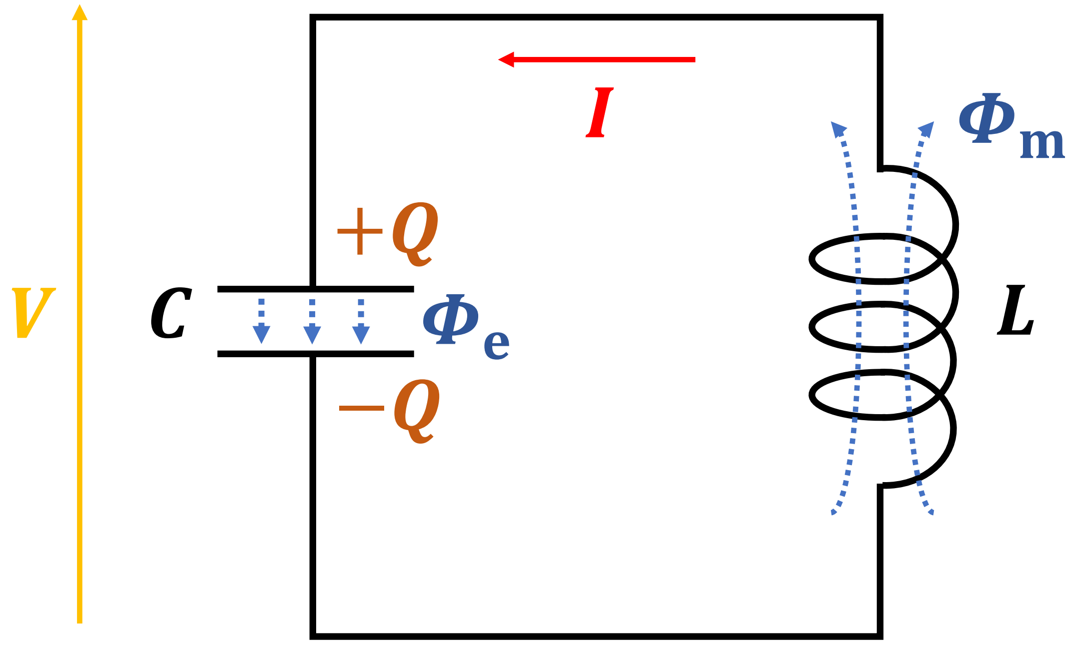
- Equations of motion:

$$\frac{dQ}{dt} = I, \qquad \frac{d\Phi_{\mathrm{m}}}{dt} = -V$$

- Oscillation:

$$L\frac{d^2Q}{dt^2} = L\frac{dI}{dt} = \frac{d\Phi_{\mathrm{m}}}{dt} = -V = -\frac{1}{C}Q$$

$$\boxed{\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q = -\omega^2 Q} \qquad \omega = \frac{1}{\sqrt{LC}}$$



Quantized LC circuit

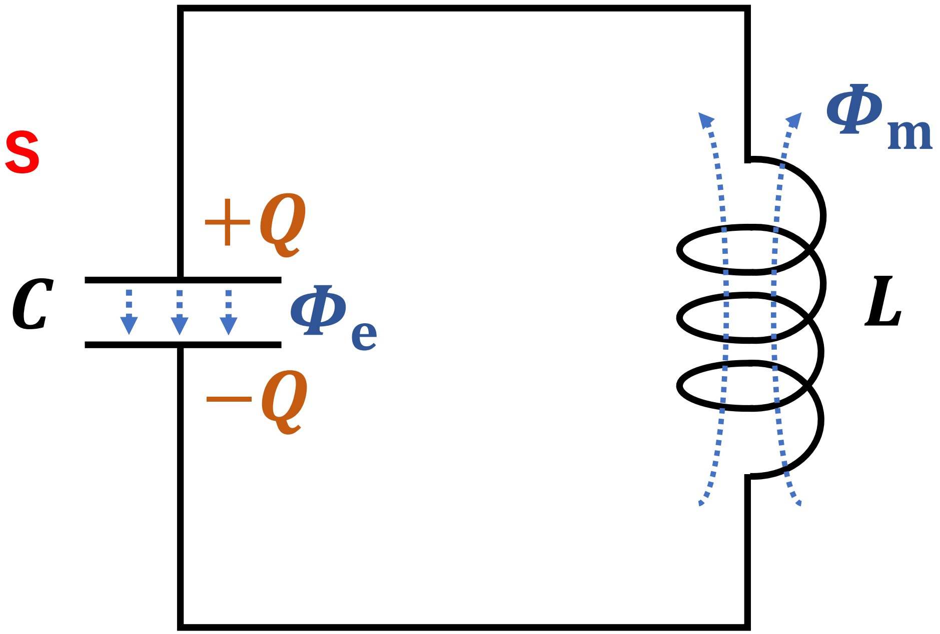
- Hamiltonian: $\hat{H} = \frac{1}{2L}\hat{\Phi}_{\mathrm{m}}^2 + \frac{1}{2C}\hat{Q}^2$
- A product of electric charge Q and magnetic flux Φ_m has a dimension of action integral $S = \int eA \cdot v\,dt = e\int A \cdot dr = e\int B \cdot n\,d\sigma$
- Assume the commutation relation $[\hat{Q}, \hat{\Phi}_{\mathrm{m}}] = i\hbar\hat{1}$.
- The Heisenberg equation reproduces the LC equation:

$$\frac{d\hat{Q}}{dt} = \frac{1}{i\hbar}[\hat{Q}, \hat{H}] = \frac{1}{L}\hat{\Phi}_{\mathrm{m}}, \qquad \frac{d\hat{\Phi}_{\mathrm{m}}}{dt} = \frac{1}{i\hbar}[\hat{\Phi}_{\mathrm{m}}, \hat{H}] = -\frac{1}{C}\hat{Q}$$

- Quantization of energy: $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),\ \hat{a} = \frac{1}{\sqrt{2\hbar\omega C}}\left(\hat{Q} + i\sqrt{\frac{C}{L}}\hat{\Phi}_m\right)$

Paradoxical problem

- Coulomb law implies $\hat{Q} = \hat{\Phi}_\mathrm{e}$ (electric charge on capacitor)
- Commutation relation: $[\hat{Q}, \hat{\Phi}_\mathrm{m}] = [\hat{\Phi}_\mathrm{e}, \hat{\Phi}_\mathrm{m}] = i\hbar\hat{1}$.
- Why can electric flux and magnetic flux be canonical conjugate non-commutative variables?
- In quantum field theory, space-likely separated two observables must be commutative, $[\hat{\Phi}_\mathrm{e}, \hat{\Phi}_\mathrm{m}] = 0$



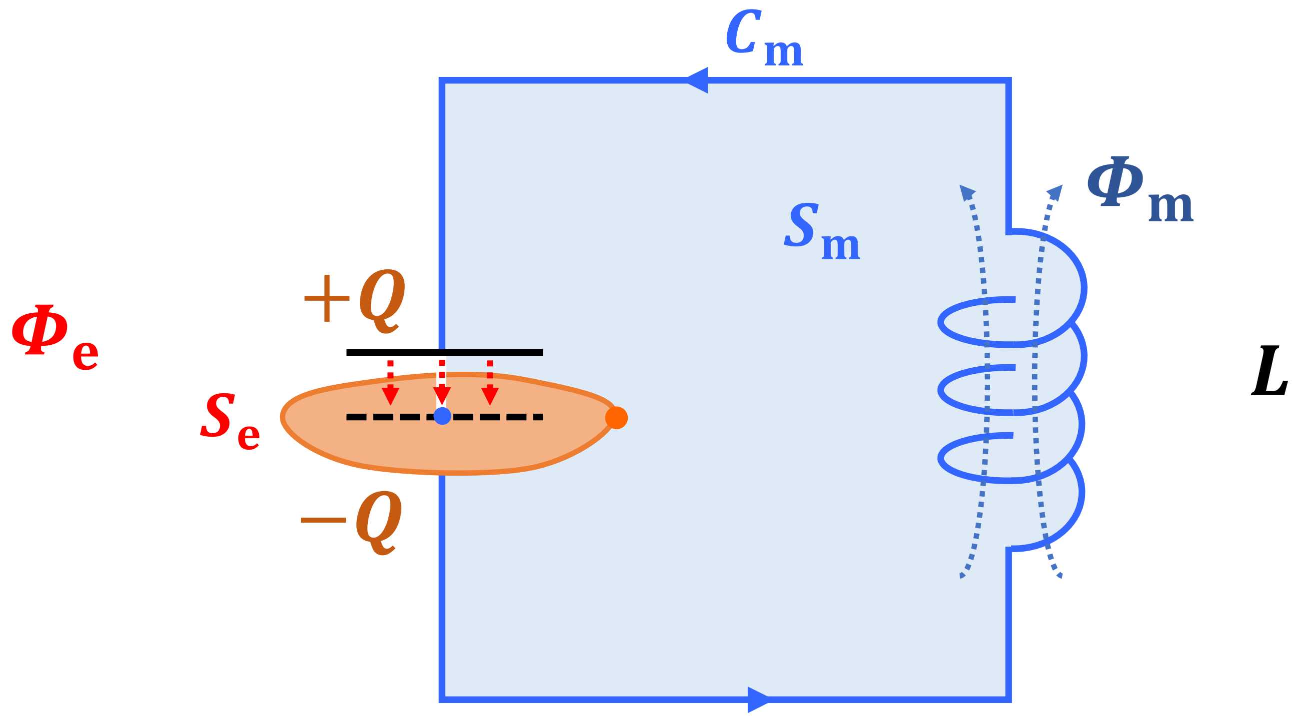
Question and Answer

Can we derive the CCR $\left[\widehat{\boldsymbol{\Phi}}_{\mathrm{e}}, \widehat{\boldsymbol{\Phi}}_{\mathrm{m}}\right] = i\hbar\widehat{\mathbf{1}}$ of electric flux and magnetic flux by a genuine quantum-field-theoretical argument?

This was the original motivation of this study and was solved.

Linking number in LC circuit

- The linking number of S_{e} and S_{m} (or C_{m}) is one in the LC circuit.
- Therefore, $[\hat{\Phi}_{\mathrm{e}}, \hat{\Phi}_{\mathrm{m}}] = [\hat{Q}, \hat{\Phi}_{\mathrm{m}}] = i\hbar\hat{1}$.
  This is the desired result.



Relativistic locality

Spatially-separated flux operators commute:

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}\right] = i\hbar\hat{\mathbf{1}}$$

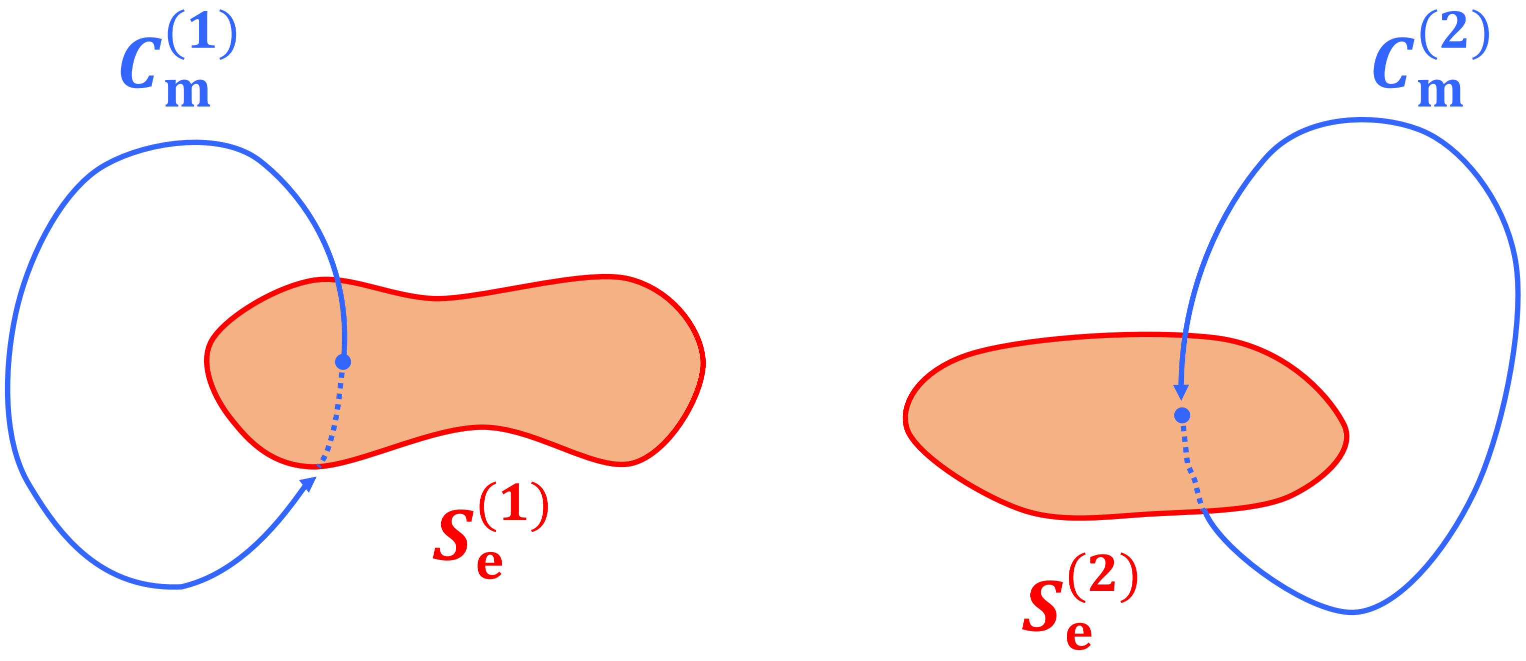
$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = i\hbar\hat{\mathbf{1}}$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = 0$$



EPR paradox

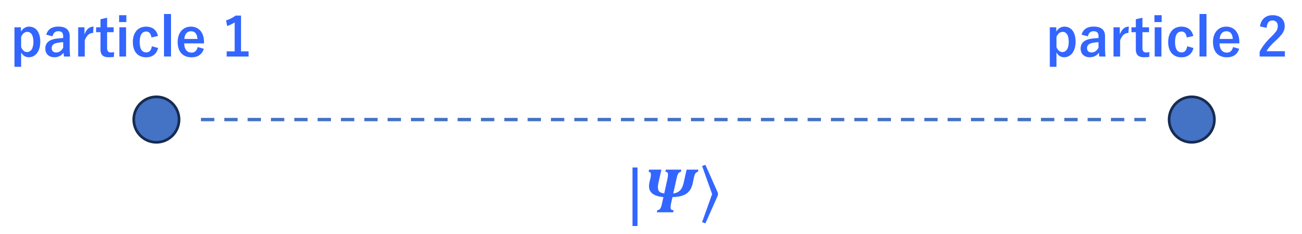
- Two particles described by $\hat{\boldsymbol{q}}^{(1)}, \hat{\boldsymbol{p}}^{(1)}, \hat{\boldsymbol{q}}^{(2)}, \hat{\boldsymbol{p}}^{(2)}$.

$$\left[\hat{\boldsymbol{q}}^{(j)}, \hat{\boldsymbol{p}}^{(k)}\right] = i\hbar\delta^{jk}\hat{\mathbf{1}}, \qquad \left[\hat{\boldsymbol{q}}^{(j)}, \hat{\boldsymbol{q}}^{(k)}\right] = \left[\hat{\boldsymbol{p}}^{(j)}, \hat{\boldsymbol{p}}^{(k)}\right] = \mathbf{0}$$

- EPR state (entangled state) $|\boldsymbol{\Psi}\rangle$ is defined by

$$\left(\hat{\boldsymbol{q}}^{(1)} - \hat{\boldsymbol{q}}^{(2)}\right)|\boldsymbol{\Psi}\rangle = \boldsymbol{D}|\boldsymbol{\Psi}\rangle, \qquad \left(\hat{\boldsymbol{p}}^{(1)} + \hat{\boldsymbol{p}}^{(2)}\right)|\boldsymbol{\Psi}\rangle = \boldsymbol{P}|\boldsymbol{\Psi}\rangle$$

($\boldsymbol{D}$ and $\boldsymbol{P}$ are c-numbers. Actually, $|\boldsymbol{\Psi}\rangle$ is an approximate eigenstate.)

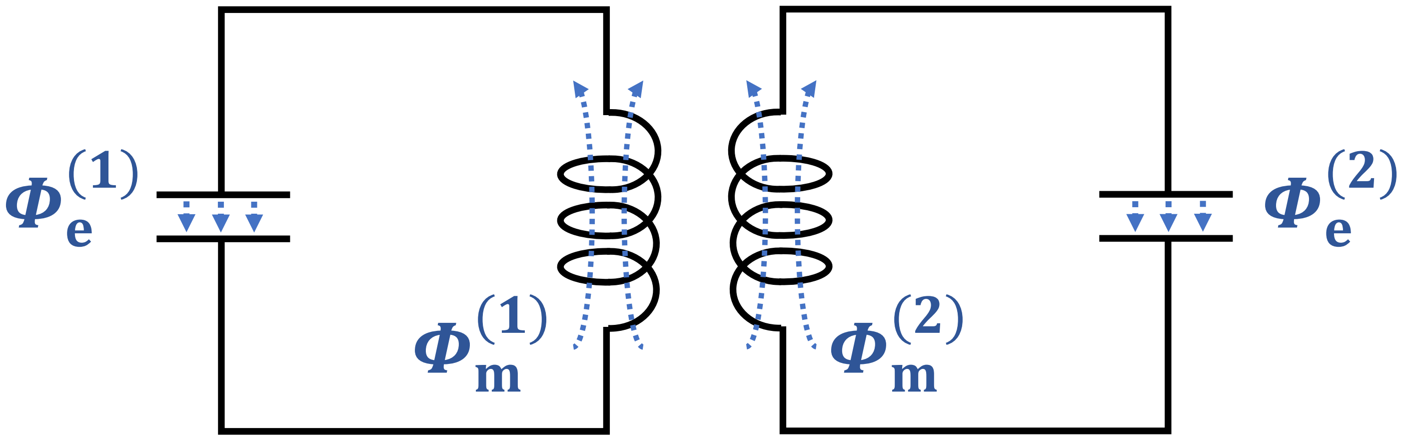


Application: a test of EPR paradox

- Coupled LC circuits provides a platform to realize the EPR state.

$$\left(\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)} - \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right)|\Psi\rangle = D|\Psi\rangle, \qquad \left(\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)} + \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right)|\Psi\rangle = P|\Psi\rangle$$

- This can provide a model for testing Clauser-Horne-Shimony-Holt inequality for continuous variables.



Summary

- Commutation relation of the electric flux and the magnetic flux
$$\left[\widehat{\boldsymbol{\Phi}}_{\mathrm{e}}, \widehat{\boldsymbol{\Phi}}_{\mathrm{m}}\right] = \boldsymbol{i\hbar}\widehat{\mathbf{1}} \cdot \boldsymbol{N}(\boldsymbol{S}_{\mathrm{e}}, \boldsymbol{C}_{\mathrm{m}})$$
is derived from quantization of electromagnetic field.
- Homological invariance of the commutator is proved.
- It is proved that spatial unlinked flux operators commute.
- Similar result has been discovered by Mikhail A. Savrov, "Commutator of Electric Charge and Magnetic Flux" (arXiv:2003.02225v2), but our argement is more detailed.
- LC circuit system can provide a platform for experimental realization of the EPR state.
- Generalization to non-abelian gauge theory is not yet done and should involve the BRST method.

Thank you for your attention.

Reference:
Homological commutation relation of electric and magnetic fluxes, 2023 *J. Phys* (open access)