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電束と磁束のホモロジー的交換関係とLC回路の量子化

Homological Commutation Relation of Electric and Magnetic Fluxes, Quantization of LC circuit

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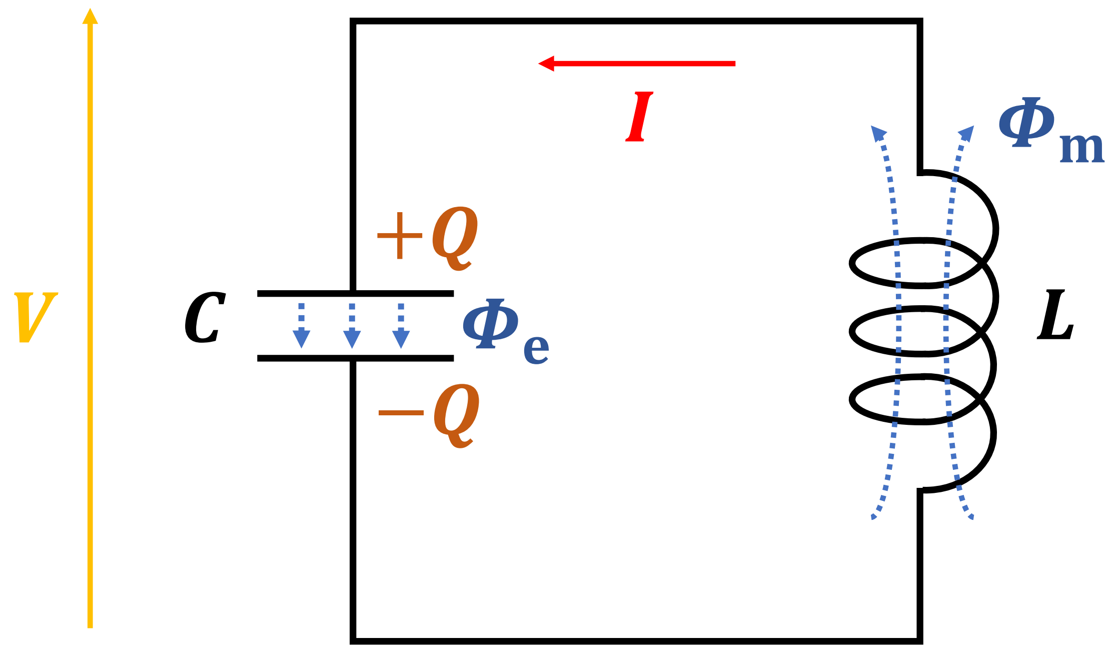
Motivation: quantization of LC circuit

- L=inductance, C=capacitance
- Inductance has a magnetic flux Φ_{m} associated with electric current I.
- Capacitance has an electric flux Φ_{e} associated with electric charge Q.



Classical LC circuit

- Constituent equations:

$$\Phi_e = Q = CV, \qquad \Phi_m = LI$$

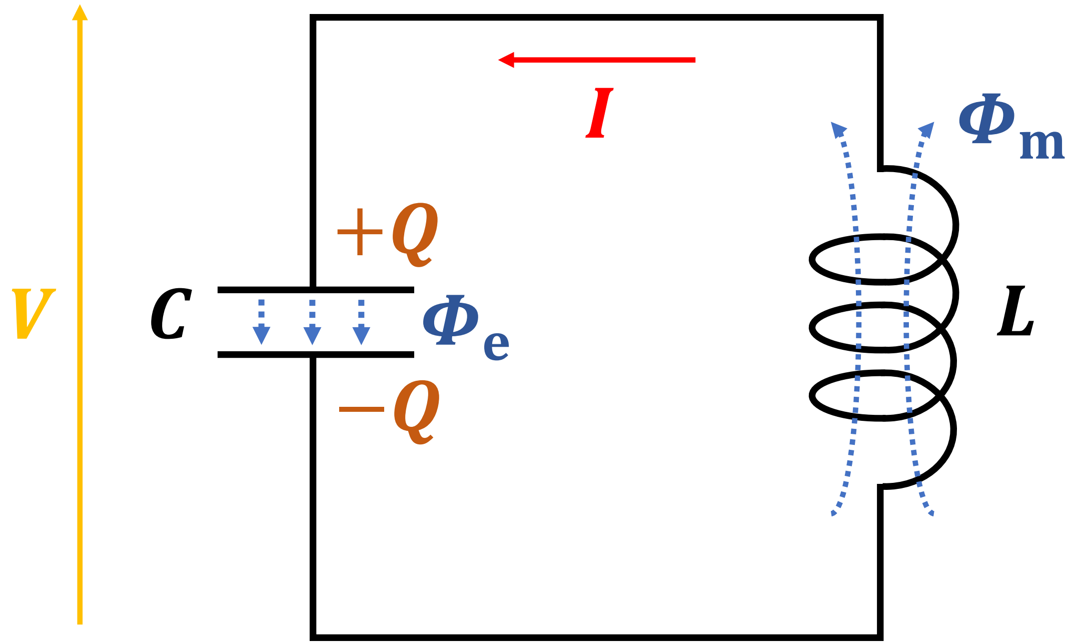
- Equations of motion:

$$\frac{dQ}{dt} = I, \qquad \frac{d\Phi_m}{dt} = -V$$

- Oscillation:

$$L\frac{d^2Q}{dt^2} = L\frac{dI}{dt} = \frac{d\Phi_m}{dt} = -V = -\frac{1}{C}Q$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q = -\omega^2 Q \qquad \omega = \frac{1}{\sqrt{LC}}$$



Phaser representation of (V, I)

$$Q = CV, \qquad \Phi_{\mathrm{m}} = LI$$

$$\frac{dQ}{dt} = C\frac{dV}{dt} = I, \qquad \frac{d\Phi_m}{dt} = L\frac{dI}{dt} = -V$$

frequency

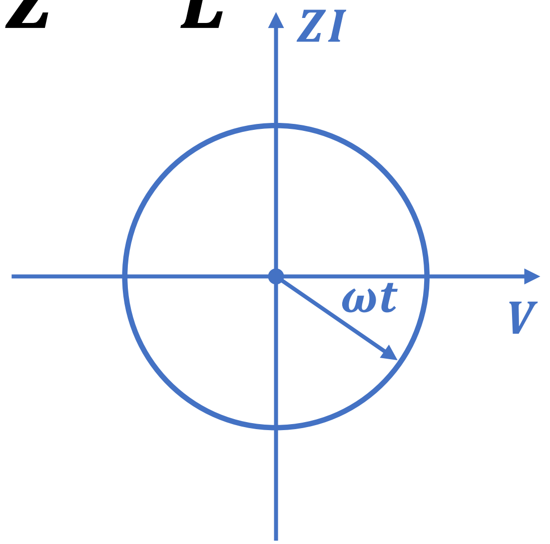
$$\omega := \frac{1}{\sqrt{LC}},$$

impedance

$$Z := \sqrt{\frac{L}{C}}, \qquad \omega Z = \frac{1}{C}, \qquad \frac{\omega}{Z} = \frac{1}{L}$$

$$\frac{dV}{dt} = \omega Z I, \qquad Z\frac{dI}{dt} = -\omega V$$

$$\frac{d}{dt}(V + iZI) = -i\omega(V + iZI)$$



Quantized LC circuit

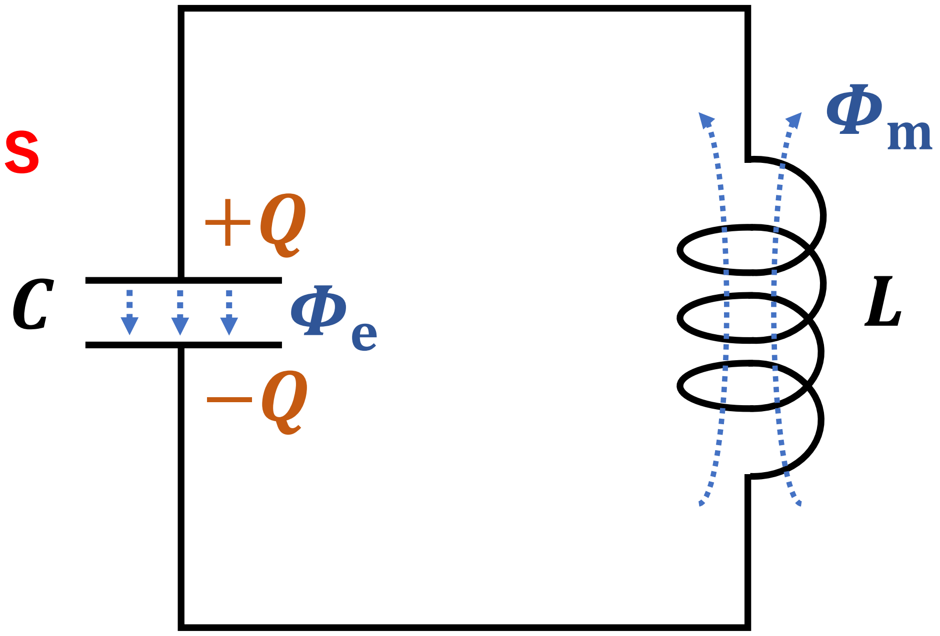
- Hamiltonian: $\hat{H} = \frac{1}{2L}\hat{\Phi}_{\mathrm{m}}^2 + \frac{1}{2C}\hat{Q}^2$
- A product of electric charge Q and magnetic flux Φ_m has a dimension of action integral $S = \int eA \cdot v\, dt = e\int A \cdot dr = e\int B \cdot n\, d\sigma$
- Assume the commutation relation $[\hat{Q}, \hat{\Phi}_{\mathrm{m}}] = i\hbar\hat{1}$.
- The Heisenberg equation reproduces the LC equation:

$$\frac{d\hat{Q}}{dt} = \frac{1}{i\hbar}[\hat{Q}, \hat{H}] = \frac{1}{L}\hat{\Phi}_m, \qquad \frac{d\hat{\Phi}_m}{dt} = \frac{1}{i\hbar}[\hat{\Phi}_m, \hat{H}] = -\frac{1}{C}\hat{Q}$$

- Quantization of energy: $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$, $\hat{a} = \frac{1}{\sqrt{2\hbar\omega C}}\left(\hat{Q} + i\sqrt{\frac{C}{L}}\hat{\Phi}_m\right)$

Question

- Coulomb law implies $\hat{Q} = \hat{\Phi}_{\mathrm{e}}$ (electric charge on capacitor)
- Commutation relation: $[\hat{Q}, \hat{\Phi}_m] = [\hat{\Phi}_e, \hat{\Phi}_m] = i\hbar\hat{1}$.
- Why can electric flux and magnetic flux be canonical conjugate non-commutative variables?
- In quantum field theory,
  space-likely separated two observables
  must be commutative, $[\hat{\Phi}_e, \hat{\Phi}_m] = 0$



“Derivation” of $[\hat{Q}, \hat{\Phi}_{\mathrm{m}}] = i\hbar\hat{1}$ a semiclassical argument (1/2)

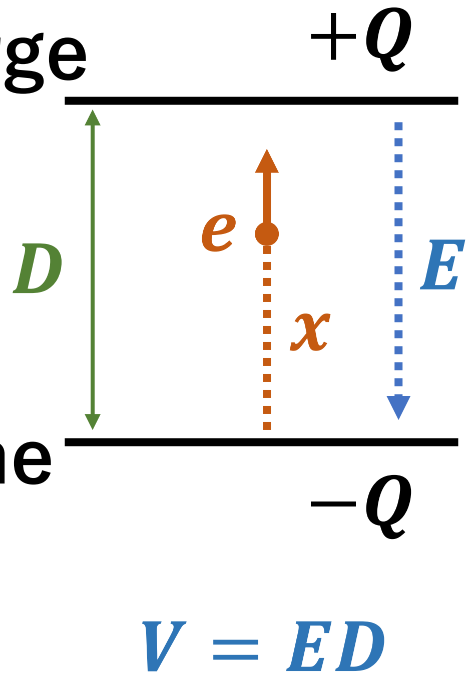
- A particle with charge e move in the space of the capacitor.
- Displacement Δx of the electric particle induces charge of the capacitor $\Delta Q = e\frac{1}{D}\Delta x$. Hence,

$$Q = \frac{e}{D}x + \mathrm{const}$$

- The momentum change of the particle is forced by the electric field that is induced by Faraday law:

$$\frac{dp}{dt} = eE = -e\frac{V}{D} = \frac{e}{D}\frac{d\Phi_{\mathrm{m}}}{dt},$$

$$p = \frac{e}{D}\Phi_m + \mathrm{const}$$



“Derivation” of $[\hat{Q}, \hat{\Phi}_\mathrm{m}] = i\hbar\hat{1}$ a semiclassical argument (2/2)

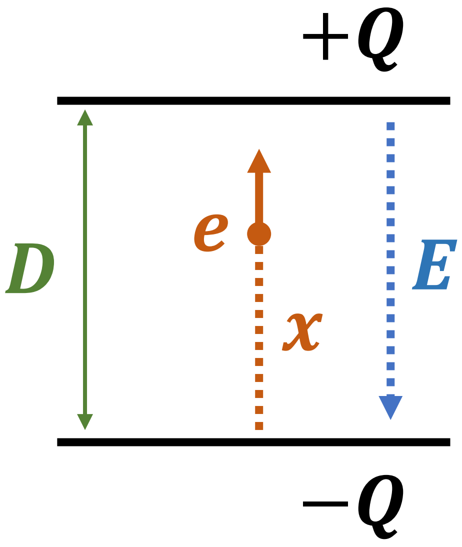
- Quantize them:

$$\hat{Q} = \frac{e}{D}\hat{x} + \mathrm{const}, \qquad \hat{p} = \frac{e}{D}\hat{\Phi}_\mathrm{m} + \mathrm{const}$$

- We get

$$[\hat{Q}, \hat{\Phi}_\mathrm{m}] = \left[\frac{e}{D}\hat{x}, \frac{D}{e}\hat{p}\right] = [\hat{x}, \hat{p}] = i\hbar\hat{1}$$

- Superficially, the canonical commutation relation (CCR) of the position and momentum of the particle leads to
the CCR of the capacitance charge and the inductance magnetic flux.



$V = ED$

Main issue

Can we derive the CCR $\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}\right] = i\hbar\hat{\mathbf{1}}$ of electric flux and magnetic flux by a genuine quantum-field-theoretical argument?

Quantum ElectroDynamics

- Decomposition of the electric field into transversal and longitudinal components: $\boldsymbol{E} = \boldsymbol{E}_{\perp} + \boldsymbol{E}_{\parallel},\ \ \mathbf{div}\,\boldsymbol{E}_{\perp} = \mathbf{0},\ \ \mathbf{rot}\,\boldsymbol{E}_{\parallel} = \mathbf{0}$
- Coulomb gauge: $\mathbf{div}\,\boldsymbol{A} = \mathbf{0},\quad \boldsymbol{\varphi}_{\mathbf{e}} = -\Delta^{-1}\boldsymbol{\rho}$
- Lagrangian: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},\ F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ (metric=$(+,-,-,-)$)
- Canonical variable:

$$\Pi_x := \frac{\partial \mathcal{L}}{\partial(\partial_0 A^x)} = F^{01} = -E_{\perp}^{x}$$

- CCR (quantization):

$$\left[\hat{A}^j(x,t), \hat{\Pi}_k(y,t)\right] = i\hbar\left(\delta_{jk} - \frac{\partial_j\partial_k}{\Delta}\right)\delta^3(x-y)$$

Derivation of $[\hat{\Phi}_e, \hat{\Phi}_m] = i\hbar\hat{1} \cdot N$

- CCR

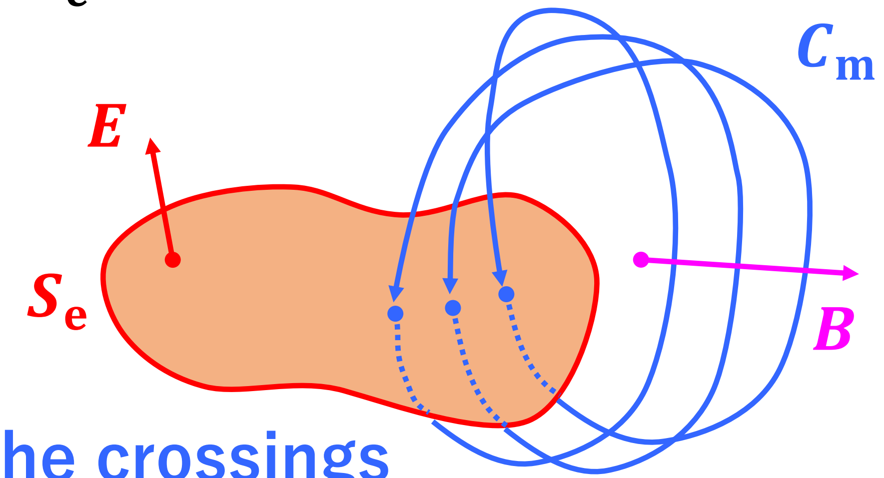
$$-\left[\hat{A}^j_\perp(x,t), \hat{E}^k_\perp(y,t)\right] = i\hbar\left(\delta_{jk} - \frac{\partial_j \partial_k}{\Delta}\right)\delta^3(x-y)$$

- Integrations:

$$\hat{\Phi}_\mathrm{m} = \int_{C_\mathrm{m}} \hat{A}_\perp \cdot dr = \int_{S_\mathrm{m}} \hat{B} \cdot n\, d\sigma, \qquad \hat{\Phi}_\mathrm{e} = \int_{S_\mathrm{e}} \hat{E}_\perp \cdot n\, d\sigma$$

- CCR yields the linking number:

$$-[\hat{\Phi}_\mathrm{m}, \hat{\Phi}_\mathrm{e}] = [\hat{\Phi}_\mathrm{e}, \hat{\Phi}_\mathrm{m}] = i\hbar\hat{1} \cdot N(S_\mathrm{e}, S_\mathrm{m})$$



N = count of the crossings

Homological invariance

- Homological deformation of surface does not change the magnetic flux since $\mathbf{div}\,\hat{\boldsymbol{B}} = \mathbf{0}$:

$$\hat{\boldsymbol{\Phi}}_m = \int_{S_{\mathrm{m}}} \hat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma} = \int_{S'_{\mathrm{m}}} \hat{\boldsymbol{B}} \cdot \boldsymbol{n}\, d\boldsymbol{\sigma}$$

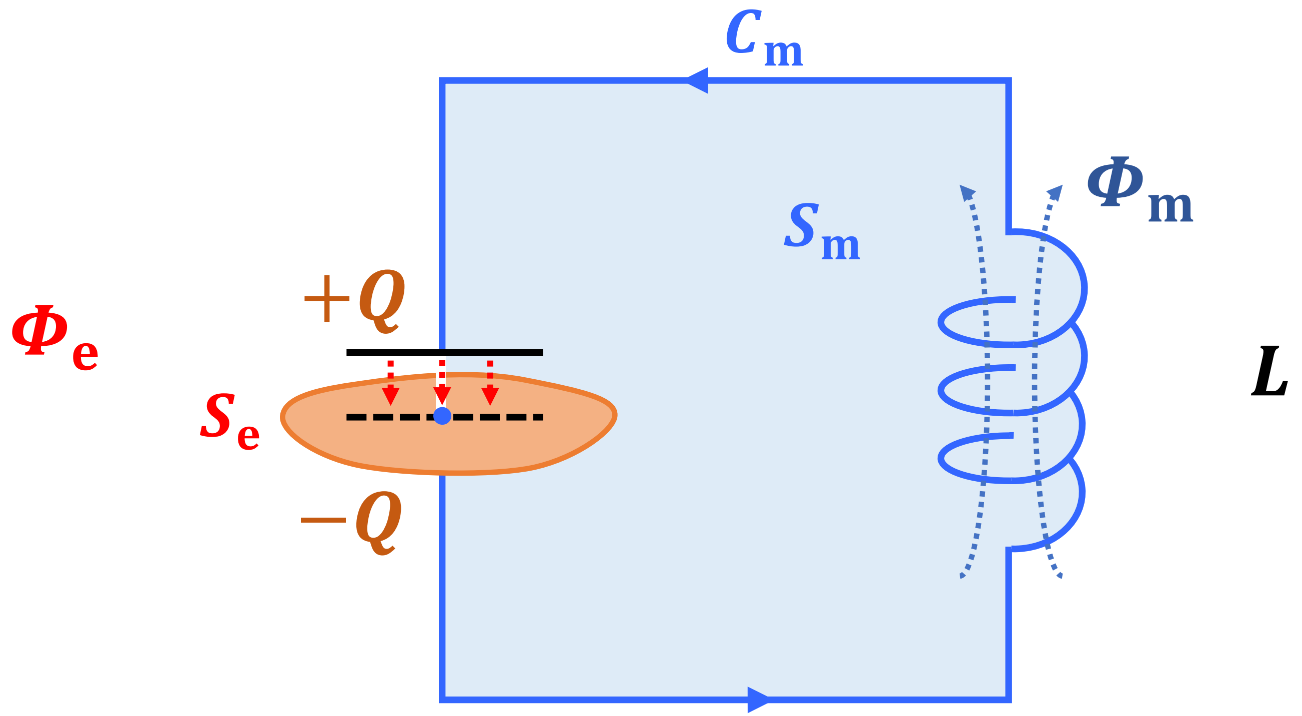
- Inclusion of the longitudinal electric field changes the electric flux but does not change the commutator since $\left[\hat{A}^j_\perp(\boldsymbol{x}, \boldsymbol{t}), \hat{E}^k_\parallel(\boldsymbol{y}, \boldsymbol{t})\right] = \mathbf{0}, \quad \left(E_\parallel = -\mathbf{grad}\,\boldsymbol{\varphi}_{\mathrm{e}} = \mathbf{grad}\,\Delta^{-1}\boldsymbol{\rho}\right)$:

$$\hat{\boldsymbol{\Phi}}_{\mathrm{e}} = \int_{S_{\mathrm{e}}} (\hat{\boldsymbol{E}}_\perp + \hat{\boldsymbol{E}}_\parallel) \cdot \boldsymbol{n}\, d\boldsymbol{\sigma} \neq \int_{S_e} \hat{\boldsymbol{E}}_\perp \cdot \boldsymbol{n}\, d\boldsymbol{\sigma}$$

- Electric-magnetic-flux commutator $\left[\hat{\boldsymbol{\Phi}}_e, \hat{\boldsymbol{\Phi}}_m\right] = i\hbar\hat{\mathbf{1}} \cdot N(S_e, S_m)$ is topologically invariant as it is to be.

Linking number in LC circuit

- The linking number of S_e and S_m (or C_m) is one in the LC circuit.
- Therefore, $[\hat{\Phi}_e, \hat{\Phi}_m] = [\hat{Q}, \hat{\Phi}_m] = i\hbar\hat{1}$.
  This is the desired result.



Relativistic locality

Spatially-separated flux operators commute:

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}\right] = i\hbar\hat{\mathbf{1}}$$

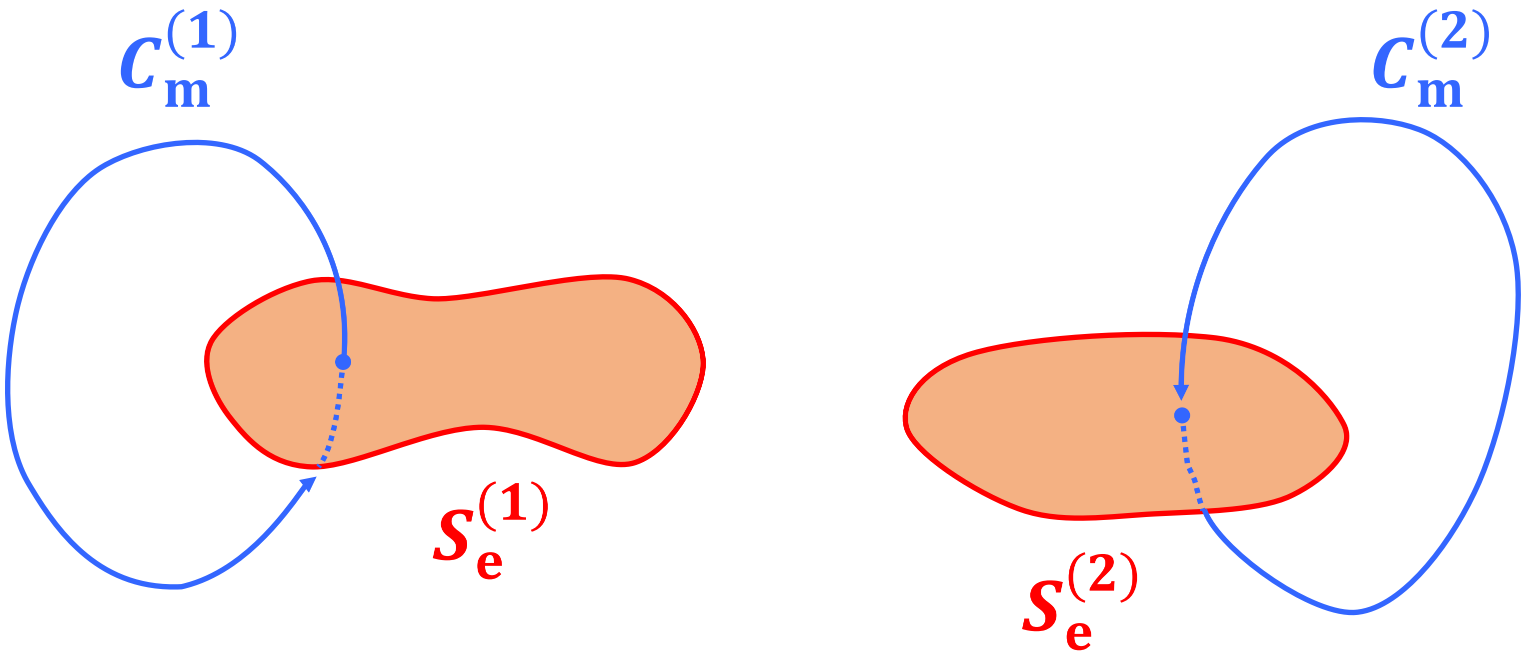
$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = i\hbar\hat{\mathbf{1}}$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right] = 0$$

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right] = 0$$



EPR paradox

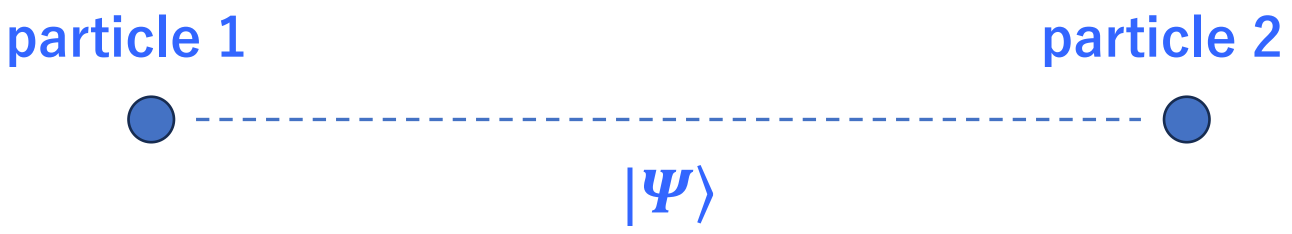
- Two particles described by $\hat{q}^{(1)}, \hat{p}^{(1)}, \hat{q}^{(2)}, \hat{p}^{(2)}$.

$$[\hat{q}^{(j)}, \hat{p}^{(k)}] = i\hbar\delta^{jk}\hat{1}, \qquad [\hat{q}^{(j)}, \hat{q}^{(k)}] = [\hat{p}^{(j)}, \hat{p}^{(k)}] = 0$$

- EPR state (entangled state) $|\Psi\rangle$ is defined by

$$\left(\hat{q}^{(1)} - \hat{q}^{(2)}\right)|\Psi\rangle = D|\Psi\rangle, \qquad \left(\hat{p}^{(1)} + \hat{p}^{(2)}\right)|\Psi\rangle = P|\Psi\rangle$$

(D and P are c-numbers. Actually, $|\Psi\rangle$ is an approximate eigenstate.)

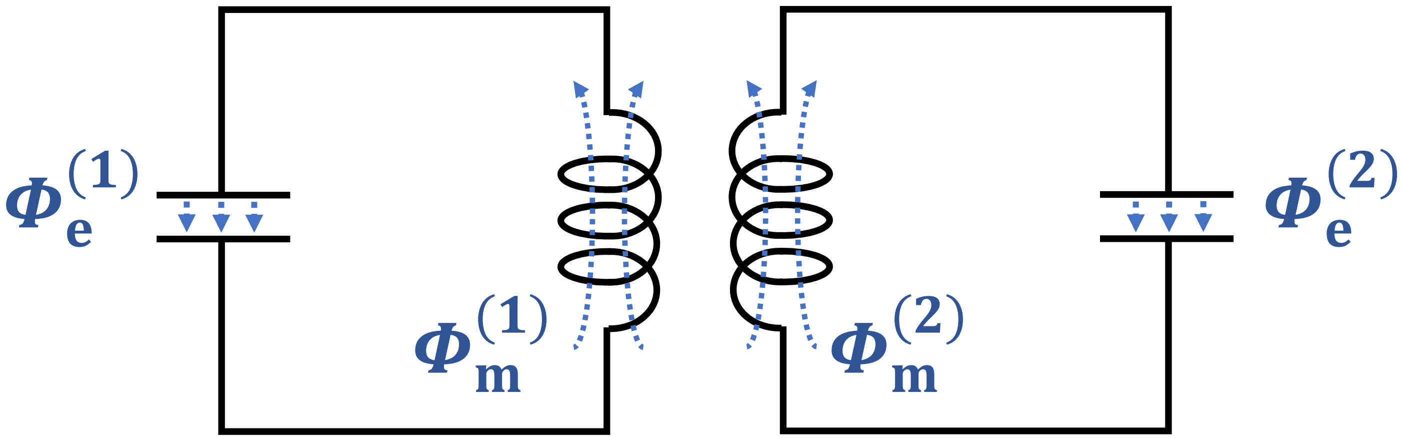


Application: a test of EPR paradox

- Coupled LC circuits provides a platform to realize the EPR state.

$$\left(\hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(1)} - \hat{\boldsymbol{\Phi}}_{\mathrm{e}}^{(2)}\right)|\Psi\rangle = D|\Psi\rangle, \qquad \left(\hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(1)} + \hat{\boldsymbol{\Phi}}_{\mathrm{m}}^{(2)}\right)|\Psi\rangle = P|\Psi\rangle$$

- This can provide a model for testing Clauser-Horne-Shimony-Holt inequality for continuous variables.



Summary

- Commutation relation of the electric flux and the magnetic flux

$$\left[\hat{\boldsymbol{\Phi}}_{\mathrm{e}}, \hat{\boldsymbol{\Phi}}_{\mathrm{m}}\right] = \boldsymbol{i}\hbar\hat{\mathbf{1}} \cdot \boldsymbol{N}(\boldsymbol{S}_{\mathrm{e}}, \boldsymbol{C}_{\mathrm{m}})$$

  is derived from QED.
- The commutator gives the linking number of the surface $\boldsymbol{S}_{\mathrm{e}}$ defining the electric flux and the loop $\boldsymbol{C}_{\mathrm{m}}$ defining magnetic flux.
- Homological invariance of the commutator is proved.
- It is proved that spatial unlinked flux operators commute.
- LC circuit system can provide a platform for experimental realization of the EPR state.
- Similar result has been discovered by Mikhail A. Savrov, “Commutator of Electric Charge and Magnetic Flux” (arXiv:2003.02225v2), but our result is more detailed.

Thank you for your attention.